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# Hardy's inequality and curvature

A. Balinsky [a], W.D. Evans [a,*], R.T. Lewis [b]

[a] *Cardiff School of Mathematics, Cardiff University, Cardiff, CF24 4AG, Wales, UK*

[b] *Department of Mathematics, University of Alabama at Birmingham, Birmingham, AL 35294-1170, USA*

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**Abstract**

A Hardy inequality of the form

$$\int_{\Omega} |\nabla f(\mathbf{x})|^p \, d\mathbf{x} \geqslant \left(\frac{p-1}{p}\right)^p \int_{\Omega} \{1 + a(\delta, \partial\Omega)(\mathbf{x})\} \frac{|f(\mathbf{x})|^p}{\delta(\mathbf{x})^p} \, d\mathbf{x},$$

for all $f \in C_0^\infty(\Omega \setminus \mathcal{R}(\Omega))$, is considered for $p \in (1, \infty)$, where $\Omega$ is a domain in $\mathbb{R}^n$, $n \geqslant 2$, $\mathcal{R}(\Omega)$ is the *ridge* of $\Omega$, and $\delta(\mathbf{x})$ is the distance from $\mathbf{x} \in \Omega$ to the boundary $\partial\Omega$. The main emphasis is on determining the dependence of $a(\delta, \partial\Omega)$ on the geometric properties of $\partial\Omega$. A Hardy inequality is also established for any doubly connected domain $\Omega$ in $\mathbb{R}^2$ in terms of a uniformization of $\Omega$, that is, any conformal univalent map of $\Omega$ onto an annulus.



*Keywords:* Hardy inequality; Distance function; Curvature; Ridge; Skeleton; Uniformization

* Corresponding author.

*E-mail addresses:* BalinskyA@cardiff.ac.uk (A. Balinsky), EvansWD@cardiff.ac.uk (W.D. Evans), rtlewis@uab.edu (R.T. Lewis).

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## 1. Introduction

This paper is a contribution to the much studied Hardy inequality

$$\int_{\Omega} \left|\nabla f(\mathbf{x})\right|^p d\mathbf{x} \geqslant c(n,p,\Omega) \int_{\Omega} \frac{|f(\mathbf{x})|^p}{\delta(\mathbf{x})^p} d\mathbf{x}, \quad f \in C_0^\infty(\Omega), \tag{1.1}$$

where $\Omega$ is a domain (an open connected set) in $\mathbb{R}^n$, $n \geqslant 2$, $1 < p < \infty$, and $\delta$ is the distance function

$$\delta(\mathbf{x}) := \operatorname{dist}\bigl(\mathbf{x}, \mathbb{R}^n \setminus \Omega\bigr) = \inf_{\mathbf{y} \in \mathbb{R}^n \setminus \Omega} |\mathbf{x} - \mathbf{y}|, \quad \mathbf{x} \in \Omega.$$

In order to put the problems we address in context and to summarize our main results, we recall some of the highlights amongst the known results to be found in the literature. For a convex domain $\Omega$ in $\mathbb{R}^n$, $n \geqslant 2$, the optimal constant in (1.1) is

$$c(n,p,\Omega) = \left(\frac{p-1}{p}\right)^p; \tag{1.2}$$

see [19] and [20]. In all cases equality is only achieved by $f = 0$. In the case $p = 2$ the inequality was improved by Brézis and Marcus in [5] to one of the form

$$\int_{\Omega} \left|\nabla f(\mathbf{x})\right|^2 d\mathbf{x} \geqslant \frac{1}{4} \int_{\Omega} \frac{|f(\mathbf{x})|^2}{\delta(\mathbf{x})^2} d\mathbf{x} + \lambda(\Omega) \int_{\Omega} \left|f(\mathbf{x})\right|^2 d\mathbf{x}, \tag{1.3}$$

where

$$\lambda(\Omega) \geqslant \frac{1}{4\operatorname{diam}(\Omega)^2}.$$

Further improvements along these lines were made in [13,8,21], including ones for $p \in (1,\infty)$ in [8] and [21], and for Hardy–Sobolev inequalities in [10]. Further pertinent references may be found in these cited papers.

For non-convex domains, a sharp constant in (1.1) is not known in general, but some sharp results were obtained in [11,6,21]. For a planar simply connected domain $\Omega$, Ancona in [1] proved the celebrated result that

$$c(2,2,\Omega) \geqslant \frac{1}{16}. \tag{1.4}$$

By assuming certain "quantifiable" degrees of convexity on a simply connected, planar domain $\Omega$, Laptev and Sobolev in [16] strengthened the Kobe *one-quarter theorem* used by Ancona in his proof and improved the lower bound in (1.4). Other results of particular relevance to the present paper are those in [3] for annular regions.

Our objective was to consider inequalities of the form

$$\int_{\Omega} |\nabla f(\mathbf{x})|^p \, d\mathbf{x} \geqslant c(n,p,\Omega) \int_{\Omega} \{1 + a(\delta, \partial\Omega)(\mathbf{x})\} \frac{|f(\mathbf{x})|^p}{\delta(\mathbf{x})^p} \, d\mathbf{x} \tag{1.5}$$

in which the function $a(\delta, \partial\Omega)$ depends on $\delta$ and geometric properties of the boundary $\partial\Omega$ of $\Omega$. We are particularly interested in domains which are either convex or have convex complements. In these cases, we determine $a(\delta, \partial\Omega)$ explicitly in terms of $\delta$ and the principal curvatures of the boundary $\partial\Omega$ of $\Omega$. Our analysis makes it necessary to consider the skeleton $\mathcal{S}(\Omega)$ and ridge $\mathcal{R}(\Omega)$ of $\Omega$: these will be defined in Section 2. A sample result is the following special case of our Corollary 2 where $\Omega$ is a convex domain with $C^2$ boundary, $p = n = 2$, and a condition on the regularity of the ridge of $\Omega$ holds (see (3.4)):

$$\int_{\Omega} |\nabla f(\mathbf{x})|^2 \, d\mathbf{x} \geqslant \frac{1}{4} \int_{\Omega} \left\{1 + \left|\frac{2\kappa\delta}{1+\kappa\delta}(\mathbf{x})\right|\right\} \frac{|f(\mathbf{x})|^2}{\delta(\mathbf{x})^2} \, d\mathbf{x}, \tag{1.6}$$

for $f \in C_0^\infty(\Omega)$, where $\kappa$ is the curvature of $\partial\Omega$. For $\Omega = B_R$, the open disc of radius $R$ and center the origin, the condition (3.4) holds, and the inversion $\mathbf{y} = \mathbf{x}/|\mathbf{x}|^2$, with $\rho = 1/R$, yields

$$\int_{\mathbb{R}^2 \setminus \overline{B_\rho}} |\nabla f(\mathbf{y})|^2 \, d\mathbf{y} \geqslant \frac{1}{4} \int_{\mathbb{R}^2 \setminus \overline{B_\rho}} \left\{-\frac{1}{|\mathbf{y}|^2} + \frac{1}{(|\mathbf{y}| - \rho)^2}\right\} |f(\mathbf{y})|^2 \, d\mathbf{y}. \tag{1.7}$$

This inequality is given in [3, Remark 1]. A significant feature of (1.6) with respect to (1.7) is that the presence of the "alien" term $-1/|\mathbf{y}|^2$ in (1.7) is explained by the curvature of the boundary. Other results in [3] are recovered from theorems in Section 3 below by taking the convex sets involved therein to be a ball. In Section 4 non-convex domains are considered. Examples are given of Hardy inequalities on a torus and on a 1-sheeted hyperboloid, which is unbounded with an unbounded interior radius.

In Theorem 7 we establish a Hardy inequality for any doubly connected domain $\Omega$ in $\mathbb{R}^2$ in terms of a uniformization of $\Omega$, i.e. any conformal univalent map of $\Omega$ onto an annulus $B_R \setminus \overline{B_\rho}$ in $\mathbb{R}^2$. This is a rich source of examples of Hardy's inequalities on non-convex domains. For example, Hardy's inequality is readily derived for the domain

$$\{z \colon \rho^2 < |\Phi(z)| < R^2\}, \quad z = x + iy,$$

where $\Phi(z) = (z-1)(z+1)$. In this case $\sqrt{\Phi(z)}$ is an appropriate uniformization.

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## 2. Curvature and distance to the boundary

The inequalities to be considered in the next section require the determination of the Laplacian of the distance function in terms of the principal curvatures of the boundary of the domain. We first recall the following facts which may be found in [7, Section 5.1]. The *skeleton* of a domain $\Omega$ is the set

$$\mathcal{S}(\Omega) := \{\mathbf{x} \in \Omega\colon\ \operatorname{card} N(\mathbf{x}) > 1\}$$

where $N(\mathbf{x}) = \{\mathbf{y} \in \partial\Omega\colon\ |\mathbf{y} - \mathbf{x}| = \delta(\mathbf{x})\}$, the set of *near points* of $\mathbf{x}$ on $\partial\Omega$. The function $\delta$ is differentiable at $\mathbf{x}$ if and only if $\mathbf{x} \notin \mathcal{S}(\Omega)$. In $\Omega \setminus \mathcal{S}(\Omega)$, $\nabla\delta$ is continuous and $\nabla\delta(\mathbf{x}) = (\mathbf{y} - \mathbf{x})/|\mathbf{y} - \mathbf{x}|$, where $N(\mathbf{x}) = \{\mathbf{y}\}$. (When $N(\mathbf{x}) = \{\mathbf{y}\}$, we sometimes abuse the notation and write $\mathbf{y} = N(\mathbf{x})$.) Since $\delta$ is Lipschitz continuous, and hence differentiable almost everywhere by Rademacher's Theorem, $\mathcal{S}(\Omega)$ is of Lebesgue measure zero. In [7, Corollary 5.1.4], it is shown that if $\mathbf{x} \in \Omega$ and $\mathbf{y} \in N(\mathbf{x})$ then $N(\mathbf{y} + t[\mathbf{x} - \mathbf{y}]) = \{\mathbf{y}\}$ for all $t \in (0, \lambda)$, where

$$\lambda := \sup\{t \in (0, \infty)\colon\ \mathbf{y} \in N(\mathbf{y} + t[\mathbf{x} - \mathbf{y}])\}.$$

The point $p(\mathbf{x}) := \mathbf{y} + \lambda[\mathbf{x} - \mathbf{y}]$ is called the *ridge point* of $\mathbf{x} \in \Omega$ and $\mathcal{R}(\Omega) := \{p(\mathbf{x})\colon\ \mathbf{x} \in \Omega\}$ is called the *ridge* of $\Omega$. For further details and properties of $\mathcal{S}(\Omega)$ and $\mathcal{R}(\Omega)$ we refer to [7, §5.1]. In particular, note that $\mathcal{R}(\Omega)$ can be much larger than $\mathcal{S}(\Omega)$ and $\mathcal{S}(\Omega) \subseteq \mathcal{R}(\Omega) \subseteq \overline{\mathcal{S}(\Omega)}$. We shall be assuming throughout, without further mention, that $\mathcal{R}(\Omega)$ is closed relative to $\Omega$, and so $\mathcal{R}(\Omega) = \overline{\mathcal{S}(\Omega)}$; it is proved in [7, Theorem 5.1.10], that this is equivalent to the functions $p$ and $\delta\ o\ p$ being continuous on $\Omega$. Note that in [18, Theorem 1.1], it is proved that if $\Omega$ has a $C^{2,1}$ boundary (cf. next paragraph) then $\delta\ o\ p$ is Lipschitz continuous as a function defined on the boundary.

Let $\Omega$ be a domain in $\mathbb{R}^n$, $n \geqslant 2$, with boundary $\partial\Omega \in C^2$: this means that locally, after a rotation of co-ordinates, $\partial\Omega$ is the graph $x_n = \phi(x_1, x_2, \ldots, x_{n-1})$ of a function $\phi \in C^2$. We consider a change of co-ordinates

$$\Gamma : (s^1, s^2, \ldots, s^n) \to \mathbf{x} = (x^1, x^2, \ldots, x^n)$$

defined for $\mathbf{x} \in \Omega$ by

$$\mathbf{x} = \boldsymbol{\gamma}(s^1, s^2, \ldots, s^{n-1}) + s^n \mathbf{n}(s^1, s^2, \ldots, s^{n-1}). \tag{2.1}$$

Here $\boldsymbol{\gamma}(s^1, s^2, \ldots, s^{n-1}) \in \partial\Omega$, and $\mathbf{n}(s^1, s^2, \ldots, s^{n-1})$ is the internal unit normal to $\partial\Omega$ at $\boldsymbol{\gamma}(s^1, s^2, \ldots, s^{n-1})$, i.e. pointing in the direction of $\mathbf{x}$. The co-ordinates $(s^1, s^2, \ldots, s^{n-1})$ are chosen with respect to principal directions through the (unique) near point $N(\mathbf{x})$ of $\mathbf{x}$ on $\partial\Omega$, such that, with $\mathbf{s}' = (s^1, s^2, \ldots, s^{n-1})$ and

$$\frac{\partial\gamma}{\partial s^i} =: \mathbf{v}_i = (v_i^1, v_i^2, \ldots, v_i^n), \quad i = 1, 2, \ldots, n-1,$$

we have

$$\begin{gathered} \langle \mathbf{v}_i, \mathbf{v}_j \rangle = \delta_{ij}, \qquad \langle \mathbf{v}_i, \mathbf{n} \rangle = 0, \\ \frac{\partial \mathbf{n}(\mathbf{s}')}{\partial s^i} = \kappa_i(\mathbf{s}') \frac{\partial\gamma(\mathbf{s}')}{\partial s^i} = \kappa_i(\mathbf{s}') \mathbf{v}_i(\mathbf{s}'), \end{gathered} \tag{2.2}$$

where $\kappa_i$, $i = 1, 2, \ldots, n-1$, are the *principal curvatures* of $\partial\Omega$ at the near point $\mathbf{y}$ of $\mathbf{x}$, and the angular notation denotes scalar product. In (2.2), the signs of the principal curvatures are determined by the direction of the normal $\mathbf{n}$. If $\Omega$ is convex, the principal curvatures of $\partial\Omega$ are non-positive, while if the domain under consideration is $\bar{\Omega}^c = \mathbb{R}^n \setminus \bar{\Omega}$, the principal curvatures are non-negative.

We set $s^n = \delta$; in (2.1), $\delta$ is equal to the distance $\delta(\mathbf{x})$ of $\mathbf{x}$ to $\partial\Omega$.

The following result (with $g(t) = t$) may be found in Gilbarg and Trudinger [12, Lemma 14.17], for points close to the $C^2$ boundary of a bounded domain $\Omega$. For our reader's convenience, we give our proof, which is designed for our needs. Note the proof of Lemma 2.2 in [17], from which it follows that if $\Omega$ has a $C^2$-boundary, then $\delta \in C^2$ on $\Omega \setminus \mathcal{R}(\Omega)$. We have seen that $\mathcal{S}(\Omega)$ is of zero measure and hence so is $\mathcal{R}(\Omega)$ if $\mathcal{S}(\Omega)$ is closed. We caution the reader that in [12] and [17] computations are made with respect to the outward unit normal (rather than the inward unit normal as in this paper) causing a different sign for the principal curvatures $\kappa_i$, $i = 1, \dots, n-1$.

**Lemma 1.** *Let $\Omega$ be a domain in $\mathbb{R}^n$, $n \geqslant 2$, with $C^2$ boundary, and set $\delta(\mathbf{x}) := \operatorname{dist}(\mathbf{x}, \partial\Omega)$. Then $\delta \in C^2(\Omega \setminus \mathcal{R}(\Omega))$, and for $g(\mathbf{x}) = g(\delta(\mathbf{x}))$, $g \in C^2(\mathbb{R}^+)$,*

$$\Delta_{\mathbf{x}} g(\mathbf{x}) = \frac{\partial^2 g}{\partial \delta^2}(\mathbf{x}) + \sum_{i=1}^{n-1} \left( \frac{\kappa_i}{1 + \delta\kappa_i} \right) \frac{\partial g}{\partial \delta}(\mathbf{x}), \tag{2.3}$$

*where the $\kappa_i$ are the principal curvatures of $\partial\Omega$ at the near point $N(\mathbf{x})$ of $\mathbf{x}$. Eq.* (2.3) *holds for all $\mathbf{x} \in \Omega \setminus \mathcal{R}(\Omega)$.*

**Proof.** From (2.1), for $i = 1, 2, \dots, n$,

$$\frac{\partial x^i}{\partial s^j} = \frac{\partial \gamma^i}{\partial s^j} + \delta \frac{\partial n^i}{\partial s^j}, \quad j = 1, 2, \dots, n-1, \qquad \frac{\partial x^i}{\partial \delta} = n^i, \tag{2.4}$$

and so, by (2.2),

$$\frac{\partial \mathbf{x}}{\partial s^j} = (1 + \delta\kappa_j)\mathbf{v}_j, \quad j = 1, 2, \dots, n-1, \qquad \frac{\partial \mathbf{x}}{\partial s^n} = \mathbf{n}. \tag{2.5}$$

Therefore, on recalling that $s^n = \delta$

$$\begin{aligned}
\begin{pmatrix} 1 & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & 1 \end{pmatrix} &= \begin{pmatrix} \frac{\partial x^1}{\partial s^1} & \cdots & \frac{\partial x^1}{\partial s^n} \\ \vdots & & \vdots \\ \frac{\partial x^n}{\partial s^1} & \cdots & \frac{\partial x^n}{\partial s^n} \end{pmatrix} \begin{pmatrix} \frac{\partial s^1}{\partial x^1} & \cdots & \frac{\partial s^1}{\partial x^n} \\ \vdots & & \vdots \\ \frac{\partial s^n}{\partial x^1} & \cdots & \frac{\partial s^n}{\partial x^n} \end{pmatrix} \\
&= \begin{pmatrix} (1+\delta\kappa_1)v_1^1 & \cdots & (1+\delta\kappa_{n-1})v_{n-1}^1 & n^1 \\ \vdots & & \vdots & \vdots \\ (1+\delta\kappa_1)v_1^n & \cdots & (1+\delta\kappa_{n-1})v_{n-1}^n & n^n \end{pmatrix} \\
&\quad \times \begin{pmatrix} \frac{\partial s^1}{\partial x^1} & \cdots & \frac{\partial s^1}{\partial x^n} \\ \vdots & & \vdots \\ \frac{\partial s^n}{\partial x^1} & \cdots & \frac{\partial s^n}{\partial x^n} \end{pmatrix}.
\end{aligned} \tag{2.6}$$

It follows from (2.2) that

$$\begin{pmatrix} (1+\delta\kappa_1)^{-1}v_1^1 & \cdots & (1+\delta\kappa_1)^{-1}v_1^n \\ \vdots & & \vdots \\ (1+\delta\kappa_{n-1})^{-1}v_{n-1}^1 & \cdots & (1+\delta\kappa_{n-1})^{-1}v_{n-1}^n \\ n^1 & \cdots & n^n \end{pmatrix} = \begin{pmatrix} \frac{\partial s^1}{\partial x^1} \cdots \frac{\partial s^1}{\partial x^n} \\ \vdots \\ \frac{\partial s^n}{\partial x^1} \cdots \frac{\partial s^n}{\partial x^n} \end{pmatrix}. \tag{2.7}$$

Therefore, for $j = 1, 2, \dots, n$, $i = 1, 2, \dots, n-1$,

$$\frac{\partial s^i}{\partial x^j} = [1+\delta\kappa_i]^{-1} v_i^j, \qquad \frac{\partial \delta}{\partial x^j} = n^j \tag{2.8}$$

and, employing the usual summation convention,

$$\frac{\partial^2 \delta}{\partial (x^j)^2} = \frac{\partial n^j}{\partial s^i}\frac{\partial s^i}{\partial x^j} = \sum_{i=1}^{n-1}[1+\delta\kappa_i]^{-1} v_i^j \frac{\partial n^j}{\partial s^i} + \frac{\partial n^j}{\partial \delta} n^j.$$

Consequently

$$\begin{aligned} \Delta\delta &= \sum_{j=1}^{n}\left\{\sum_{i=1}^{n-1}[1+\delta\kappa_i]^{-1} v_i^j \frac{\partial n^j}{\partial s^i} + n^j \frac{\partial n^j}{\partial \delta}\right\} \\ &= \sum_{i=1}^{n-1}[1+\delta\kappa_i]^{-1}\left\langle \mathbf{v}_i, \frac{\partial \mathbf{n}}{\partial s^i}\right\rangle + \left\langle \mathbf{n}, \frac{\partial \mathbf{n}}{\partial \delta}\right\rangle \\ &= \sum_{i=1}^{n-1}\kappa_i[1+\delta\kappa_i]^{-1} \end{aligned} \tag{2.9}$$

by (2.2). From the Chain Rule, we have

$$\frac{\partial g}{\partial x^j} = \frac{\partial g}{\partial s^i}\frac{\partial s^i}{\partial x^j} + \frac{\partial g}{\partial \delta}\frac{\partial \delta}{\partial x^j} = \frac{\partial g}{\partial \delta}\frac{\partial \delta}{\partial x^j}$$

and, on using (2.8),

$$\begin{aligned} \Delta_{\mathbf{x}} g &= \frac{\partial}{\partial s^k}\left[\frac{\partial g}{\partial \delta}\frac{\partial \delta}{\partial x^j}\right]\frac{\partial s^k}{\partial x^j} = \left[\frac{\partial^2 g}{\partial s^k \partial \delta} n^j + \frac{\partial g}{\partial \delta}\frac{\partial n^j}{\partial s^k}\right]\frac{\partial s^k}{\partial x^j} \\ &= \frac{\partial^2 g}{\partial(\delta^2)} + \frac{\partial g}{\partial \delta}\sum_{j=1}^{n}\sum_{k=1}^{n-1}\kappa_k v_k^j [1+\delta\kappa_k]^{-1} v_k^j \\ &= \frac{\partial^2 g}{\partial(\delta^2)} + \frac{\partial g}{\partial \delta}\sum_{k=1}^{n-1}\kappa_k[1+\delta\kappa_k]^{-1}. \end{aligned}$$

The lemma is therefore proved. □

**Remark 1.** It follows from (2.8) that the terms $[\kappa_i/(1+\delta\kappa_i)](\mathbf{y})$, $\mathbf{y}=N(\mathbf{x})$, $i=1,2,\ldots,n-1$, in (2.10), are the principal curvatures of the level surface of $\delta$ through $\mathbf{x}$ at $\mathbf{x}$. Furthermore, $\frac{1}{n-1}\sum_{i=1}^{n-1}[\kappa_i/(1+\delta\kappa_i)](\mathbf{y})$ is the mean curvature of this level surface at $\mathbf{x}$.

**Remark 2.** If $\Omega$ is convex, then $S(\bar{\Omega}^c)=R(\bar{\Omega}^c)=\emptyset$.

**Remark 3.** If $\Omega$ is a convex domain with a $C^2$-boundary, we have noted that the principal curvatures of $\partial\bar{\Omega}^c$ are non-negative and hence

$$\Delta\delta(\mathbf{x})=\tilde{\kappa}(\mathbf{y}):=\sum_{i=1}^{n-1}\big[\kappa_i/(1+\delta\kappa_i)\big](\mathbf{y})\geqslant 0 \quad \text{for all } \mathbf{x}\in\bar{\Omega}^c \tag{2.10}$$

where $\{\mathbf{y}\}=N(\mathbf{x})$.

We claim that for $\Omega$ convex, we also have

$$\Delta\delta(\mathbf{x})=\tilde{\kappa}(\mathbf{y}):=\sum_{i=1}^{n-1}\big[\kappa_i/(1+\delta\kappa_i)\big](\mathbf{y})\leqslant 0 \quad \text{for all } \mathbf{x}\in\Omega\setminus\mathcal{R}(\Omega). \tag{2.11}$$

To see this, let $\mathbf{x}_0$ be an arbitrary point in $\Omega$, $\mathbf{y}_0=N(\mathbf{x}_0)$, and let $h(s)$ be a principal curve through $\mathbf{y}_0$ on $\partial\Omega$ with curvature $\kappa$ at $\mathbf{y}_0$: thus

$$h(0)=\mathbf{y}_0,\qquad \big|h'(0)\big|=1,\qquad h''(0)=-\kappa\mathbf{n}(0),$$

where $\mathbf{n}$ is the **inward normal** to $\partial\Omega$ at $\mathbf{y}_0$. The function

$$f(s):=\big|h(s)-\mathbf{x}_0\big|^2$$

has a minimum at $s=0$ and so at $s=0$, we have

$$f'(s)=2h'(s)\cdot\big[h(s)-\mathbf{x}_0\big]=0,$$
$$f''(s)=2h''(s)\cdot\big[h(s)-\mathbf{x}_0\big]+2\big|h'(s)\big|^2\geqslant 0.$$

Consequently, since $[h(0)-\mathbf{x}_0]=-\delta\mathbf{n}$, we have

$$1+\delta\kappa\geqslant 0.$$

This is true for all principal directions and so, since the principal curvatures are non-positive, our claim (2.11) is established.

**Remark 4.** If $\kappa_i(\mathbf{y})\geqslant 0$, then

$$1+\delta(\mathbf{x})\kappa_i(\mathbf{y})\geqslant 1,\qquad N(\mathbf{x})=\{\mathbf{y}\}. \tag{2.12}$$

Therefore, if $\Omega$ is the complement of a closed convex domain with $C^2$ boundary, then (2.12) holds for all $i$ throughout $\Omega$, since then $\mathcal{R}(\Omega)=\mathcal{S}(\Omega)=\emptyset$.

Suppose $\Omega$ is convex and $\partial\Omega \in C^2$. Then for all $\mathbf{x} \in \Omega \setminus \mathcal{R}(\Omega)$ and for all $i$

$$1+\delta(\mathbf{x})\kappa_i(\mathbf{y}) = 1-\delta(\mathbf{x})\big|\kappa_i(\mathbf{y})\big| > 0, \qquad N(\mathbf{x}) = \{\mathbf{y}\}. \tag{2.13}$$

This is proved as follows. We saw in Remark 3 that $1+\delta\kappa_i \geqslant 0$. Suppose for some $i$ and some $\mathbf{x} \in \Omega$ with $N(\mathbf{x}) = \{\mathbf{y}\}$, that $\delta(\mathbf{x}) = 1/|\kappa_i(\mathbf{y})|$. If $\mathbf{x}$ lies outside $\mathcal{R}(\Omega)$, then it follows from [7, Corollary 5.1.4], that there exists a point $\mathbf{w} \in \Omega \setminus \mathcal{R}(\Omega)$ on the ray from $\mathbf{y}$ through $\mathbf{x}$ such that $N(\mathbf{w}) = \{\mathbf{y}\}$ and $\delta(\mathbf{w}) > \delta(\mathbf{x})$. But, this would imply that $1+\delta(\mathbf{w})\kappa_i(\mathbf{y}) < 0$ which is a contradiction.

**Remark 5.** Suppose $\Omega$ is convex with $\partial\Omega \in C^2$. Then $\mathbf{x} \in \mathcal{R}(\Omega) \setminus \mathcal{S}(\Omega)$ if and only if for $N(\mathbf{x}) = \{\mathbf{y}\}$

$$1+\delta(\mathbf{x})\kappa_i(\mathbf{y}) = 0, \quad \text{for some } i. \tag{2.14}$$

The ridge in this case has zero Lebesgue measure.

## 3. Inequalities inside and outside domains

We first establish the following general inequality:

**Theorem 1.** *Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, be a domain having a ridge $\mathcal{R}(\Omega)$ and a sufficiently smooth boundary for Green's formula to hold. Let $\delta(\mathbf{x}) = \operatorname{dist}(\mathbf{x}, \mathbb{R}^n \setminus \Omega)$. Then for all $f \in C_0^\infty(\Omega \setminus \mathcal{R}(\Omega))$ and $p \in (1,\infty)$,*

$$\int_\Omega |\nabla\delta \cdot \nabla f|^p \, d\mathbf{x} \geqslant \left(\frac{p-1}{p}\right)^p \int_\Omega \left\{1 - \frac{p\delta\Delta\delta}{p-1}\right\} \frac{|f|^p}{\delta^p} \, d\mathbf{x}. \tag{3.1}$$

**Proof.** For any vector field $V$ we have the identity

$$\int_\Omega (\operatorname{div} V)|f|^p \, d\mathbf{x} = -p\left[\operatorname{Re}\int_\Omega (V\cdot\nabla f)|f|^{p-2}\overline{f} \, d\mathbf{x}\right] \tag{3.2}$$

for all $f \in C_0^\infty(\Omega \setminus \mathcal{R}(\Omega))$. Choose

$$V = -p\nabla\delta/\delta^{p-1}.$$

Then, for any $\varepsilon > 0$,

$$\begin{aligned}\int_\Omega \operatorname{div} V|f|^p \, d\mathbf{x} &\leqslant p^2\left(\int_\Omega |\nabla\delta\cdot\nabla f|^p \, d\mathbf{x}\right)^{1/p}\left(\int_\Omega \frac{|f|^p}{\delta^p} \, d\mathbf{x}\right)^{1-1/p} \\ &\leqslant p\varepsilon^p \int_\Omega |\nabla\delta\cdot\nabla f|^p \, d\mathbf{x} + p(p-1)\varepsilon^{-p/(p-1)} \int_\Omega \frac{|f|^p}{\delta^p} \, d\mathbf{x}\end{aligned}$$

which gives, since $\operatorname{div} V = (p-1)p\delta^{-p} - p\delta^{1-p}\Delta\delta$ for $\mathbf{x} \in \Omega \setminus \mathcal{R}(\Omega)$,

$$\int_\Omega |\nabla\delta \cdot \nabla f|^p \, d\mathbf{x} \geqslant \varepsilon^{-p} \int_\Omega \left[(p-1) - (p-1)\varepsilon^{-p/(p-1)} - \delta\Delta\delta\right] \frac{|f|^p}{\delta^p} \, d\mathbf{x}.$$

The proof of (3.1) is completed on choosing $\varepsilon = [p/(p-1)]^{\frac{(p-1)}{p}}$. $\square$

**Corollary 1.** *If $\partial\Omega \in C^2$, then for all $f \in C_0^\infty(\Omega \setminus \mathcal{R}(\Omega))$*

$$\int_\Omega |\nabla\delta \cdot \nabla f|^p \, d\mathbf{x} \geqslant \left(\frac{p-1}{p}\right)^p \int_\Omega \left\{1 - \frac{p\delta\tilde{\kappa}}{p-1}\right\} \frac{|f|^p}{\delta^p} \, d\mathbf{x} \tag{3.3}$$

*where $\tilde{\kappa} := \sum_{i=1}^{n-1} \frac{\kappa_i}{1+\delta\kappa_i}$.*

**Proof.** The proof follows from Lemma 1 and Theorem 1. $\square$

In many cases, we are able to prove an inequality for all $f \in C_0^\infty(\Omega)$.

**Theorem 2.** *Assume the hypothesis of Theorem* 1. *Let $\mathcal{R}(\Omega)$ be the intersection of a decreasing family of open neighborhoods $\{S_\epsilon\colon \epsilon > 0\}$ with smooth boundaries, and let $\eta_\epsilon(\mathbf{x})$ denote the unit inward normal at $\mathbf{x} \in \partial S_\epsilon$. If*

$$(\nabla\delta \cdot \eta_\epsilon)(\mathbf{x}) \geqslant 0, \quad \mathbf{x} \in \partial S_\epsilon \tag{3.4}$$

*for all $\epsilon$ sufficiently small and*

$$\frac{p-1}{p} \geqslant [\delta\Delta\delta](\mathbf{x}), \quad \mathbf{x} \in \Omega \setminus R(\Omega), \tag{3.5}$$

*then* (3.1) *holds for all $f \in C_0^\infty(\Omega)$.*

**Proof.** We proceed as in (3.2), but now with $f \in C_0^\infty(\Omega)$, and account for the contribution of the boundary of $S_\epsilon$. On using (3.4) we have for all $f \in C_0^\infty(\Omega)$

$$\int_{\Omega\setminus S_\epsilon} (\operatorname{div} V)|f|^p \, d\mathbf{x} \leqslant -p\left[\operatorname{Re} \int_{\Omega\setminus S_\epsilon} (V \cdot \nabla f)|f|^{p-2}\bar{f} \, d\mathbf{x}\right]. \tag{3.6}$$

On proceeding as in the proof of Theorem 1 we obtain

$$\begin{aligned}\int_\Omega |\nabla\delta \cdot \nabla f|^p \, d\mathbf{x} &\geqslant \int_{\Omega\setminus S_\epsilon} |\nabla\delta \cdot \nabla f|^p \, d\mathbf{x} \\ &\geqslant \left(\frac{p-1}{p}\right)^p \int_\Omega \left[1 - \frac{p}{p-1}\delta\Delta\delta\right] \frac{|f|^p}{\delta^p} \chi_{\Omega\setminus S_\epsilon} \, d\mathbf{x}.\end{aligned}$$

The proof concludes on using (3.5) and the monotone convergence theorem. $\square$

**Corollary 2.** *Suppose that the hypothesis of Theorem* 2 *is satisfied and that $\Omega$ is convex with a $C^2$ boundary. Then for all $f \in C_0^\infty(\Omega)$*

$$\int_\Omega |\nabla\delta \cdot \nabla f|^p \, d\mathbf{x} \geqslant \left(\frac{p-1}{p}\right)^p \int_\Omega \left\{1 + \frac{p\delta|\tilde{\kappa}|}{p-1}\right\} \frac{|f|^p}{\delta^p} \, d\mathbf{x}. \tag{3.7}$$

**Proof.** Since $\Omega$ is convex, it follows from Lemma 1 that $-\Delta\delta = -\tilde{\kappa} \geqslant 0$ for $\mathbf{x} \in \Omega \setminus \mathcal{R}(\Omega)$; see Remark 3. Therefore, (3.5) must hold and the result then follows from Theorem 2. □

**Corollary 3.** *Let $\Omega$ be a ball $B_R := \{\mathbf{x} \in \mathbb{R}^n\colon\ |\mathbf{x}| < R\}$. Then for $p > 1$,*

$$\int_{B_R} |\nabla f|^p \, d\mathbf{x} - \left(\frac{p-1}{p}\right)^p \int_{B_R} \frac{|f|^p}{\delta^p} \, d\mathbf{x} \geqslant \left(\frac{p-1}{p}\right)^{p-1} \int_{B_R} \frac{(n-1)|f|^p}{|\mathbf{x}|\delta^{p-1}} \, d\mathbf{x} \tag{3.8}$$

*for all $f \in C_0^\infty(B_R)$.*

**Proof.** In this case we have that $\mathcal{R}(B_R) = \overline{\mathcal{S}(B_R)} = \{0\}$ and $\delta = R - |\mathbf{x}|$. We now have that $S_\epsilon = B_\epsilon$ and on $\partial S_\epsilon$, $\nabla\delta = -\frac{\mathbf{x}}{|\mathbf{x}|} = \eta_\epsilon$. Therefore (3.4) holds implying that (3.8) is valid since $|\tilde{\kappa}| = (n-1)/|\mathbf{x}|$. □

In [9, Theorem 3.1], it is proved that for $\Omega \subset \mathbb{R}^n$ convex,

$$\int_\Omega |\nabla u|^2 \, d\mathbf{x} - \frac{1}{4} \int_\Omega \frac{|u|^2}{\delta^2} \, d\mathbf{x} \geqslant c_\alpha D_{int}^{-(\alpha+2)} \int_\Omega \delta^\alpha |u|^2 \, d\mathbf{x}, \tag{3.9}$$

for any $\alpha > -2$, where

$$c_\alpha = \begin{cases} 2^\alpha(2\alpha+3), & \text{if } \alpha \geqslant -1, \\ 2^\alpha(\alpha+2)^2, & \text{if } \alpha \in (-2,-1) \end{cases}$$

and $D_{int} := 2\sup\{\delta(\mathbf{x})\colon\ \mathbf{x} \in \Omega\}$. A comparison of the right-hand side of (3.9), when $\Omega = B_R$, with that in the case $p = 2$ of (3.8), is now made to seek further evidence of the significance of the curvature in these inequalities. Set $\alpha = -2 + \varepsilon$. Then the terms to be compared from (3.8) and (3.9), respectively, are $I_1 = (n-1)/2\delta(\mathbf{x})|\mathbf{x}|$ and $I_2 = c_\alpha D_{int}^{-(\alpha+2)} \delta^\alpha(\mathbf{x})$, with $\delta(\mathbf{x}) = R - |\mathbf{x}|$. It is readily shown that

$$I_1 - I_2 \geqslant \begin{cases} \frac{2n-1-2\varepsilon}{4\delta|\mathbf{x}|}, & \text{if } \varepsilon \geqslant 1, \\ \frac{(2n-2)R-(2n-1)|\mathbf{x}|}{4\delta^2|\mathbf{x}|}, & \text{if } 0 < \varepsilon < 1. \end{cases}$$

A similar comparison can be made in the $L^p$ case using Theorem 3.2 of [9] with $p = q$ and $\alpha > -p$. Also see [10].

**Example 1** *(The infinite cylinder).* Let $\Omega = B_1(0) \times \mathbb{R}$, where $B_1(0)$ is the unit ball, center the origin, in $\mathbb{R}^2$. Clearly, $\Omega$ is convex and $R(\Omega)$ is the $z$-axis. The distance function is $\delta = 1 - \sqrt{x^2 + y^2}$,

$$\nabla\delta = -(x, y, 0)(1-\delta)^{-1}, \qquad \eta = -(x, y, 0)/(1-\delta) \quad \text{on } \partial S_\epsilon,$$

and $\Delta\delta = \frac{-1}{1-\delta}$, where $S_\epsilon := \{\mathbf{x} = (x, y, z) \in \Omega\colon\ x^2 + y^2 < \epsilon\}$. Therefore, (3.7) holds for this cylinder with $|\tilde{\kappa}| = |\Delta\delta| = \frac{1}{1-\delta}$.

**Theorem 3.** *Under the conditions of Theorem* 1, *we have for all $f \in C_0^\infty(\Omega \setminus \mathcal{R}(\Omega))$,*

$$\int_\Omega \left|\nabla f(\mathbf{x})\right|^2 d\mathbf{x} \geqslant \frac{1}{4}\int_\Omega \left\{\frac{(n-2)^2}{|\mathbf{x}|^2} + \frac{(1+2|\delta\Delta\delta|)}{\delta^2} + 2(n-2)\frac{\mathbf{x}\cdot\nabla\delta}{|\mathbf{x}|^2\delta}\right\}\left|f(\mathbf{x})\right|^2 d\mathbf{x}. \tag{3.10}$$

*In particular, if $\Omega$ is a convex domain with a $C^2$ boundary, the conditions of the theorem are met and $\Delta\delta = \sum_{i=1}^{n-1} \kappa_i/(1+\delta\kappa_i)$ in $\Omega \setminus \mathcal{R}(\Omega)$.*

**Proof.** Let

$$V(\mathbf{x}) = -2\frac{\nabla\delta(\mathbf{x})}{\delta(\mathbf{x})} + 2(n-2)\frac{\mathbf{x}}{|\mathbf{x}|^2}.$$

Then

$$\operatorname{div} V(\mathbf{x}) = \frac{2}{\delta(\mathbf{x})^2} - \frac{2\Delta\delta(\mathbf{x})}{\delta(\mathbf{x})} + \frac{2(n-2)^2}{|\mathbf{x}|^2} \geqslant 0$$

and

$$\frac{1}{4}\left|V(\mathbf{x})\right|^2 = \frac{1}{\delta^2} + \frac{(n-2)^2}{|\mathbf{x}|^2} - 2(n-2)\frac{\mathbf{x}\cdot\nabla\delta}{|\mathbf{x}|^2\delta}.$$

For any $\varepsilon > 0$, and $f \in C_0^\infty(\Omega \setminus \mathcal{R}(\Omega))$

$$\begin{aligned}
\int_\Omega (\operatorname{div} V)|f|^2\, d\mathbf{x} &= -2\mathrm{Re}\left[\int_\Omega (V\cdot\nabla f)\overline{f}\, d\mathbf{x}\right] \\
&\leqslant 2\left(\int_\Omega |\nabla f|^2\, d\mathbf{x}\right)^{1/2}\left(\int_\Omega |V|^2|f|^2\, d\mathbf{x}\right)^{1/2} \\
&\leqslant \varepsilon^2\int_\Omega |\nabla f|^2\, d\mathbf{x} + \varepsilon^{-2}\int_\Omega |V|^2|f|^2\, d\mathbf{x}.
\end{aligned}$$

The result follows on choosing $\varepsilon = 2$. $\square$

When $\Omega = B_R$, we have on substituting in Theorem 3, $\delta(\mathbf{x}) = R - |\mathbf{x}|$, $\kappa_i(\mathbf{x}) = -1/R$, $i = 1, 2, \ldots, n-1$, and so $\Delta\delta(\mathbf{x}) = -(n-1)/|\mathbf{x}|$ from Lemma 1 (or by direct calculation),

$$\int_{B_R} \left|\nabla f(\mathbf{x})\right|^2 d\mathbf{x} \geqslant \frac{1}{4}\int_{B_R} \left\{\frac{(n-2)^2}{|\mathbf{x}|^2} + \frac{1}{\delta(\mathbf{x})^2} + \frac{2}{|\mathbf{x}|\delta(\mathbf{x})}\right\}\left|f(\mathbf{x})\right|^2 d\mathbf{x} \tag{3.11}$$

which is given in Corollary 2 in [3]. Note that (3.11) is valid for all $f \in C_0^\infty(B_R)$ – see Corollary 3.

The application of Lemma 1 to Theorem 1 also yields the following Hardy inequality in the complement of a closed convex domain. Recall that in this case $\mathcal{R}(\mathbb{R}^n \setminus \bar{\Omega}) = \emptyset$.

**Theorem 4.** *Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, be convex with a $C^2$ boundary. Then for all $f \in C_0^\infty(\mathbb{R}^n \setminus \bar{\Omega})$,*

$$\int_{\mathbb{R}^n\setminus\bar{\Omega}} \left|\nabla f(\mathbf{x})\right|^p d\mathbf{x} \geqslant \left(\frac{p-1}{p}\right)^p \int_{\mathbb{R}^n\setminus\bar{\Omega}} \left\{1 - \frac{p\tilde{\kappa}\delta}{p-1}\right\}\frac{|f(\mathbf{x})|^p}{\delta(\mathbf{x})^p}\, d\mathbf{x}, \tag{3.12}$$

*where $\tilde{\kappa} = \sum_{i=1}^{n-1} \frac{\kappa_i}{1+\delta\kappa_i} \geqslant 0$.*

Note that if $\Omega = B_\rho$, the integrand on the right-hand side of (3.12) is non-negative if and only if

$$|\mathbf{x}| \leqslant 2\frac{(n-1)\rho}{2n-3}.$$

The following is another form of Hardy inequality, reminiscent of that derived in [4, Theorem 3.1].

**Theorem 5.** *Let $\Omega$ be a convex domain in $\mathbb{R}^n$ with a $C^2$ boundary. Then for all $f \in C_0^\infty(\mathbb{R}^n \setminus \bar{\Omega})$,*

$$\int_{\mathbb{R}^n\setminus\bar{\Omega}} \delta^p |\nabla\delta \cdot \nabla f|^p\, d\mathbf{x} \geqslant \frac{1}{p^p}\int_{\mathbb{R}^n\setminus\bar{\Omega}} [1 + p\tilde{\kappa}\delta]|f|^p\, d\mathbf{x},$$

*where $\tilde{\kappa} = \sum_{i=1}^{n-1} \frac{\kappa_i}{1+\delta\kappa_i} \geqslant 0$.*

**Proof.** The proof follows the lines of that of Theorem 3.1 in [4]. From (3.2),

$$\begin{aligned}\int_{\mathbb{R}^n\setminus\bar{\Omega}} (\operatorname{div} V)|f|^p\, d\mathbf{x} &\leqslant p\biggl(\int_{\mathbb{R}^n\setminus\bar{\Omega}} |V \cdot \nabla f|^p\, d\mathbf{x}\biggr)^{1/p}\biggl(\int_{\mathbb{R}^n\setminus\bar{\Omega}} |f|^p\, d\mathbf{x}\biggr)^{(p-1)/p} \\ &\leqslant \varepsilon^p \int_{\mathbb{R}^n\setminus\bar{\Omega}} |V \cdot \nabla f|^p\, d\mathbf{x} + (p-1)\varepsilon^{-p/(p-1)}\int_{\mathbb{R}^n\setminus\bar{\Omega}} |f|^p\, d\mathbf{x}.\end{aligned}$$

On choosing $V = \delta^2$, we have

$$2^p \epsilon^p \int_{\mathbb{R}^n \setminus \bar{\Omega}} \delta^p |\nabla\delta \cdot \nabla f|^p \, d\mathbf{x} + (p-1)\epsilon^{-\frac{p}{p-1}} \int_{\mathbb{R}^n \setminus \bar{\Omega}} |f|^p \, d\mathbf{x} \geqslant \int_{\mathbb{R}^n \setminus \bar{\Omega}} \left[\Delta\delta^2\right] |f|^p \, d\mathbf{x}$$

$$= 2 \int_{\mathbb{R}^n \setminus \bar{\Omega}} [1 + \tilde{\kappa}\delta] \, d\mathbf{x} \tag{3.13}$$

by Lemma 1. Hence, as in (3.6) of [4],

$$2^p \int_{\mathbb{R}^n \setminus \bar{\Omega}} \delta^p |\nabla\delta \cdot \nabla f|^p \, d\mathbf{x} \geqslant K(\epsilon) \int_{\mathbb{R}^n \setminus \bar{\Omega}} |f|^p \, d\mathbf{x} + 2\epsilon^{-p} \int_{\mathbb{R}^n \setminus \bar{\Omega}} \tilde{\kappa}\delta |f|^p \, d\mathbf{x}$$

where

$$K(\epsilon) = 2\epsilon^{-p} - (p-1)\epsilon^{-\frac{p^2}{p-1}}$$

has a maximum value of $(2/p)^p$ at $\epsilon = (p/2)^{(p-1)/p}$. The proof is completed by making the substitution for this value of $\epsilon$. □

When $p = 2$, it is readily shown that the substitution $u = \delta f$ in Theorem 5 yields (3.12).

**Example 2.** If $\Omega = B_\rho$ in Theorem 5, then

$$\int_{\mathbb{R}^n \setminus \overline{B_\rho}} \left(|\mathbf{x}| - \rho\right)^p |\nabla f|^p \, d\mathbf{x} \geqslant \frac{1}{p^p} \int_{\mathbb{R}^n \setminus \overline{B_\rho}} \left[1 + p(n-1)\frac{|\mathbf{x}| - \rho}{|\mathbf{x}|}\right] |f|^p \, d\mathbf{x}$$

for all $f \in C_0^\infty(\mathbb{R}^n \setminus \overline{B_\rho})$.

We have the following analogue of Theorem 1 in [3] for an annulus bounded by convex domains.

**Theorem 6.** *Let $\Omega_1$, $\Omega_2$, be convex domains in $\mathbb{R}^n$, $n \geqslant 2$, with $C^2$ boundaries and $\bar{\Omega}_1 \subset \Omega_2$. For $\mathbf{x} \in \Omega := \Omega_2 \setminus \bar{\Omega}_1$ denote the distances of $\mathbf{x}$ to $\partial\Omega_1$, $\partial\Omega_2$ by $\delta_1$, $\delta_2$, respectively. Then for all $f \in C_0^\infty(\Omega \setminus \mathcal{R}(\Omega))$*

$$\begin{aligned}
\int_{\Omega_2 \setminus \bar{\Omega}_1} \left|\nabla f(\mathbf{x})\right|^2 d\mathbf{x} \geqslant \frac{1}{4} \int_{\Omega_2 \setminus \bar{\Omega}_1} \biggl\{ & \frac{(n-1)(n-3)}{|\mathbf{x}|^2} + \frac{1}{\delta_1^2} + \frac{1}{\delta_2^2} \\
& - \frac{2\Delta\delta_1}{\delta_1} - \frac{2\Delta\delta_2}{\delta_2} - \frac{2\nabla\delta_1 \cdot \nabla\delta_2}{\delta_1\delta_2} \\
& + 2(n-1)\frac{\mathbf{x} \cdot \nabla\delta_1}{|\mathbf{x}|^2\delta_1} + 2(n-1)\frac{\mathbf{x} \cdot \nabla\delta_2}{|\mathbf{x}|^2\delta_2} \biggr\} \left|f(\mathbf{x})\right|^2 d\mathbf{x}.
\end{aligned} \tag{3.14}$$

**Proof.** The starting point is again

$$\int_{\Omega_2\setminus\bar{\Omega}_1} (\operatorname{div} V)\left|f(\mathbf{x})\right|^2 d\mathbf{x} \leqslant \varepsilon^2 \int_{\Omega_2\setminus\bar{\Omega}_1} |\nabla f|^2 \, d\mathbf{x} + \varepsilon^{-2} \int_{\Omega_2\setminus\bar{\Omega}_1} |V|^2 |f|^2 \, d\mathbf{x}.$$

Guided by the proof of Corollary 1 in [3], the theorem follows on setting

$$V = 2(n-1)\frac{\nabla|\mathbf{x}|}{|\mathbf{x}|} - 2\frac{\nabla\delta_1}{\delta_1} - 2\frac{\nabla\delta_2}{\delta_2}$$

and $\varepsilon = 2$. □

If $\Omega_1 = B_\rho$, $\Omega_2 = B_R$, $R > \rho$, we have

$$\Delta\delta_1 = \frac{n-1}{|\mathbf{x}|}, \qquad \Delta\delta_2 = -\frac{n-1}{|\mathbf{x}|}$$

by Lemma 1, and $\nabla\delta_1 = -\nabla\delta_2 = \mathbf{x}/|\mathbf{x}|$. On substituting in (3.14), we derive Corollary 1 in [3], namely,

$$\int_{B_R\setminus B_\rho} \left|\nabla f(\mathbf{x})\right|^2 d\mathbf{x} \geqslant \frac{1}{4} \int_{B_R\setminus B_\rho} \left\{\frac{(n-1)(n-3)}{|\mathbf{x}|^2} + \frac{1}{\delta_1^2} + \frac{1}{\delta_2^2} + \frac{2}{\delta_1\delta_2}\right\} \left|f(\mathbf{x})\right|^2 d\mathbf{x}, \tag{3.15}$$

where $\delta_1(\mathbf{x}) = |\mathbf{x}| - \rho$, $\delta_2(\mathbf{x}) = R - |\mathbf{x}|$.

## 4. Non-convex domains

### 4.1. Torus

We show that Theorem 2 can be applied to give a Hardy-type inequality on a torus.

**Corollary 4.** *Let $\Omega \subset \mathbb{R}^3$ be the interior of a ring torus with minor radius $r$ and major radius $R > 2r$. Then $\Delta\delta < 0$ in $\Omega \setminus \mathcal{R}(\Omega)$ and*

$$\int_\Omega |\nabla\delta \cdot \nabla f|^p \, d\mathbf{x} \geqslant \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|f|^p}{\delta^p} \, d\mathbf{x} + \left(\frac{p-1}{p}\right)^{p-1} \int_\Omega \left(\frac{1}{(r-\delta)} - \frac{1}{\sqrt{x_1^2 + x_2^2}}\right) \frac{|f|^p}{\delta^{p-1}} \, d\mathbf{x} \tag{4.1}$$

*for all $f \in C_0^\infty(\Omega)$, where $\mathbf{x} \in \Omega$ has co-ordinates $(x_1, x_2, x_3)$, and the last integrand is positive.*

**Proof.** The domain $\Omega$ under consideration is the "doughnut-shaped" domain generated by rotating a disc of radius $r$ about a co-planar axis at a distance $R$ from the center of the disc. The fact that $\Delta\delta \leqslant 0$ was proved by D.H. Armitage and Ü. Kuran [2]. We give a different proof here which meets our purposes using a curvature argument.

The ridge of the torus is

$$\mathcal{R}(\Omega) = \{\mathbf{x}\colon\ \rho(\mathbf{x}) = 0\},$$

where $\rho(\mathbf{x})$ is the distance from the point $\mathbf{x}$ in $\Omega$ to the center of the cross-section and $\delta(\mathbf{x}) = r - \rho(\mathbf{x})$. Moreover, in the notation of Theorem 2,

$$S_\epsilon = \{\mathbf{x}\colon\ \rho(\mathbf{x}) < \epsilon\},$$

and points on the surface of $S_\epsilon$ are on the level surface $\rho(\mathbf{x}) = \epsilon$, so that the unit inward normal to $\partial S_\varepsilon$ is $\eta_\varepsilon = -\nabla\rho(\mathbf{x})/|\nabla\rho(\mathbf{x})| = \nabla\delta$. Therefore $\nabla\delta\cdot\nabla\eta > 0$.

For $\mathbf{x} \in \Omega \setminus \mathcal{R}(\Omega)$, let $\mathbf{y} = N(\mathbf{x}) = (y_1, y_2, y_3)$ have the parametric co-ordinates

$$\begin{aligned} y_1 &= \left(R + r\cos s^2\right)\cos s^1, \\ y_2 &= \left(R + r\cos s^2\right)\sin s^1, \\ y_3 &= r\sin s^2, \end{aligned}$$

where $s^1, s^2 \in (-\pi, \pi]$. The principal curvatures at $\mathbf{y} \in \partial\Omega$ are

$$\kappa_1 = -\frac{1}{r}, \qquad \kappa_2 = -\frac{\cos s^2}{R + r\cos s^2},$$

e.g., see Kreyszig [15, p. 135], and so, by Lemma 1,

$$\begin{aligned} \Delta\delta(\mathbf{x}) = \sum_{i=1}^{2}\left(\frac{\kappa_i}{1+\delta\kappa_i}\right)(\mathbf{y}) &= -\frac{R + 2(r-\delta)\cos s^2}{(r-\delta)(R + (r-\delta)\cos s^2)} \\ &= -\frac{\sqrt{x_1^2 + x_2^2} + (r-\delta)\cos s^2}{(r-\delta)\sqrt{x_1^2 + x_2^2}} < 0 \end{aligned}$$

since $R + r\cos s^2 = \sqrt{x_1^2 + x_2^2} + \delta(\mathbf{x})\cos s^2$ and $R > 2r$. The inequality (4.1) follows from Theorem 2. $\square$

### 4.2. 1-sheeted hyperboloid

Next, we apply Theorem 1 to the 1-sheeted hyperboloid

$$\Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3\colon\ x_1^2 + x_2^2 < 1 + x_3^2\}. \tag{4.2}$$

This is non-convex and unbounded with infinite volume and infinite interior diameter $D_{int}(\Omega)$. To calculate the principal curvatures, we choose the following parametric co-ordinates for $\mathbf{y} \in \partial\Omega$:

$$
\begin{aligned}
y_1(s,t) &= \sqrt{s^2+1}\cos t, \\
y_2(s,t) &= \sqrt{s^2+1}\sin t, \\
y_3(s,t) &= s,
\end{aligned}
$$

for $t \in [0, 2\pi)$ and $s \in (-\infty, \infty)$. A calculation then gives (see [15, p. 132])

$$
\kappa_1 = -\frac{1}{[2s^2+1]^{3/2}}, \qquad \kappa_2 = \frac{1}{\sqrt{2s^2+1}},
$$

and if $\mathbf{y} = N(\mathbf{x})$, $\mathbf{x} \in \Omega \setminus \mathcal{R}(\Omega)$, then by Lemma 1,

$$
\Delta\delta(\mathbf{x}) = \tilde{\kappa} := \sum_{i=1}^{2} \frac{\kappa_i}{1+\delta\kappa_i} = -\frac{1}{w^3-\delta} + \frac{1}{w+\delta}, \tag{4.3}
$$

where $w = \sqrt{2s^2+1}$ is the distance of $\mathbf{y}$ from the origin, and the ridge is $\mathcal{R}(\Omega) = \{(x_1, x_2, x_3)\colon x_1 = x_2 = 0,\ x_3 \in (-\infty, \infty)\}$. Therefore $\Delta\delta(\mathbf{x})$ changes sign in $\Omega$.

To find $\mathbf{y} = N(\mathbf{x})$, we first determine the vector normal to $\partial\Omega$ at $\mathbf{y}$, namely

$$
\begin{aligned}
\mathbf{y}_s \times \mathbf{y}_t &= \begin{vmatrix} i & j & k \\ \frac{s}{\sqrt{s^2+1}}\cos t & \frac{s}{\sqrt{s^2+1}}\sin t & 1 \\ -\sqrt{s^2+1}\sin t & \sqrt{s^2+1}\cos t & 0 \end{vmatrix} \\
&= \left[-\sqrt{s^2+1}\cos t\right]i + \left[-\sqrt{s^2+1}\sin t\right]j + sk.
\end{aligned}
$$

The inward unit normal vector at $\mathbf{y}$ is therefore

$$
\mathbf{n} = \left\{\left[-\sqrt{s^2+1}\cos t\right]i + \left[-\sqrt{s^2+1}\sin t\right]j + sk\right\}/\sqrt{2s^2+1}.
$$

The distance from $\mathbf{y}$ to the ridge point $p(\mathbf{x})$ of $\mathbf{x}$ (see Section 2) is given by $\sqrt{s^2+1}/\cos\theta$, where $\cos\theta = (\mathbf{z}\cdot\mathbf{n})/|\mathbf{z}|$, and

$$
\mathbf{z} = \left[-\sqrt{s^2+1}\cos t\right]i + \left[-\sqrt{s^2+1}\sin t\right]j.
$$

Hence

$$
\sqrt{s^2+1}/\cos\theta = \sqrt{2s^2+1} = w.
$$

Consequently, the near point of $\mathbf{x}$ is the point on the boundary of $\Omega$ which is equidistant from the ridge point $p(\mathbf{x})$ of $\mathbf{x}$ and the origin.

We therefore have from Theorem 1:

**Corollary 5.** *Let $\Omega \subset \mathbb{R}^3$ be the* 1*-sheeted hyperboloid* (4.2)*. Then, for all $f \in C_0^\infty(\Omega \setminus \mathcal{R}(\Omega))$,*

$$\int_\Omega |\nabla\delta \cdot \nabla f|^p \, d\mathbf{x} \geqslant \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|f|^p}{\delta^p} \, d\mathbf{x} - \left(\frac{p-1}{p}\right)^{p-1} \int_\Omega \tilde{\kappa} \frac{|f|^p}{\delta^{p-1}} \, d\mathbf{x}, \tag{4.4}$$

*where $\tilde{\kappa}$ is given in* (4.3)*, with $w = |\mathbf{y}| = \delta(p(\mathbf{x}))$, $\mathbf{y} = N(\mathbf{x})$ and $p(\mathbf{x})$ the ridge point of* $\mathbf{x}$.

## *4.3. Doubly connected domains*

A domain $\Omega \subset \mathbb{R}^2 \equiv \mathbb{C}$ is *doubly connected* if its boundary is a disjoint union of 2 simple curves. If it has a smooth boundary then it can be mapped conformally onto an annulus $\Omega_{\rho,R} = B_R \setminus \overline{B_\rho} = \{z \in \mathbb{C}\colon\ \rho < |z| < R\}$, for some $\rho$, $R$; see [22, Theorem 1.2].

**Lemma 2.** *Let $\Omega_1 \subset \Omega_2 \subset \mathbb{C}$ and $B_\rho \subset B_R \subset \mathbb{C}$, $0 < \rho < R$, where $B_r$ is the disc of radius $r$ centered at the origin. Let*

$$F : \Omega_2 \setminus \bar{\Omega}_1 \to B_R \setminus \overline{B_\rho}$$

*be analytic and univalent. Then for $\mathbf{z} = x_1 + i x_2$, $\mathbf{x} = (x_1, x_2) \in \Omega_2 \setminus \bar{\Omega}_1$,*

$$\mathfrak{F}(\mathbf{z}) := -\frac{|F'(\mathbf{z})|^2}{|F(\mathbf{z})|^2} + \left|F'(\mathbf{z})\right|^2 \left\{\frac{1}{|F(\mathbf{z})| - \rho} + \frac{1}{R - |F(\mathbf{z})|}\right\}^2 \tag{4.5}$$

*is invariant under scaling, rotation, and inversion. Hence, $\mathfrak{F}$ does not depend on the choice of the mapping $F$, but only on the geometry of $\Omega_2 \setminus \bar{\Omega}_1$.*

**Proof.** The fact that $\mathfrak{F}$ is invariant under scaling and rotations is straightforward. To see that it is also invariant under inversions suppose that $F(\mathbf{z}) = 1/G(\mathbf{z})$. Then, under inversion $\mathfrak{F}(\mathbf{z})$ becomes

$$\begin{aligned}
&-\frac{|G'(\mathbf{z})|^2}{|G(\mathbf{z})|^2} + \frac{|G'(\mathbf{z})|^2}{|G(\mathbf{z})|^4} \left\{\frac{1}{\frac{1}{|G(\mathbf{z})|} - \rho^{-1}} + \frac{1}{R^{-1} - \frac{1}{|G(\mathbf{z})|}}\right\}^2 \\
&= -\frac{|G'(\mathbf{z})|^2}{|G(\mathbf{z})|^2} + \frac{|G'(\mathbf{z})|^2}{|G(\mathbf{z})|^2} \left\{\frac{\rho}{\rho - |G(\mathbf{z})|} + \frac{R}{|G(\mathbf{z})| - R}\right\}^2 \\
&= -\frac{|G'(\mathbf{z})|^2}{|G(\mathbf{z})|^2} + \frac{|G'(\mathbf{z})|^2}{|G(\mathbf{z})|^2} \left\{\frac{(\rho - R)|G(\mathbf{z})|}{(\rho - |G(\mathbf{z})|)(|G(\mathbf{z})| - R)}\right\}^2 \\
&= -\frac{|G'(\mathbf{z})|^2}{|G(\mathbf{z})|^2} + \left|G'(\mathbf{z})\right|^2 \left\{\frac{1}{\rho - |G(\mathbf{z})|} + \frac{1}{|G(\mathbf{z})| - R}\right\}^2
\end{aligned}$$

implying that $\mathfrak{F}$ is invariant under inversions. The rest of the lemma follows from [14, p. 133]. $\square$

In applying the last lemma we regard $\Omega_1$, $\Omega_2$ as domains in $\mathbb{R}^2$ with $\mathbf{z} = x + iy$ and $\mathbf{x} = (x, y)$.

**Theorem 7.** *For* $\Omega := \Omega_2 \setminus \bar{\Omega}_1 \subset \mathbb{R}^2$,

$$\int_\Omega \left|\nabla u(\mathbf{x})\right|^2 d\mathbf{x} \geqslant \frac{1}{4} \int_\Omega \mathfrak{F}(\mathbf{x}) \left|u(\mathbf{x})\right|^2 d\mathbf{x}.$$

**Proof.** From Corollary 1 of [3] it follows that for all $u \in H_0^1(B_R \setminus \overline{B_\rho})$,

$$\int_{B_R \setminus \overline{B_\rho}} \left|\nabla u(\mathbf{y})\right|^2 d\mathbf{y} \geqslant \frac{1}{4} \int_{B_R \setminus \overline{B_\rho}} \left[\frac{-1}{|\mathbf{y}|^2} + \left(\frac{1}{\delta_\rho(\mathbf{y})} + \frac{1}{\delta_R(\mathbf{y})}\right)^2\right] \left|u(\mathbf{y})\right|^2 d\mathbf{y},$$

where $\delta_\rho(\mathbf{y}) := |\mathbf{y}| - \rho$ and $\delta_R(\mathbf{y}) := R - |\mathbf{y}|$. Let $F : \Omega \to \Omega_{\rho,R}$ be analytic and univalent, and set $\mathbf{y} = F(\mathbf{x})$, with $\mathbf{y} = (y_1, y_2)$, $\mathbf{x} = (x_1, x_2)$. Then, with $F'$ denoting the complex derivative,

$$d\mathbf{y} = \left|\det\left(\frac{\partial(y_1, y_2)}{\partial(x_1, x_2)}\right)\right| d\mathbf{x} = \left|F'(\mathbf{x})\right|^2 d\mathbf{x},$$

$$\nabla_{\mathbf{x}} u = \nabla_{\mathbf{y}} u \left[\frac{\partial(y_1, y_2)}{\partial(x_1, x_2)}\right]^t,$$

implying that

$$|\nabla_{\mathbf{x}} u|^2 = |\nabla_{\mathbf{y}} u|^2 \left|F'(\mathbf{x})\right|^2.$$

The theorem follows from Lemma 2. □

**Example 3.** Let $\Phi(z) = (z-1)(z+1)$ and

$$\Omega = \left\{z\colon\ \rho^2 < \left|\Phi(z)\right| < R^2\right\}$$

for $0 < \rho < R$. The function $F(z) = \sqrt{\Phi(z)}$ is analytic and univalent in $\Omega$ and

$$F : \Omega \to \Omega_{\rho,R}.$$

A calculation gives

$$\mathfrak{F}(z) = -\frac{|z|^2}{|z^2-1|^2} + \frac{|z|^2}{|z^2-1|} \frac{(R-\rho)^2}{(\sqrt{|z|^2-1} - \rho)^2 (R - \sqrt{|z|^2-1})^2}.$$

Finally, we refer the reader to further developments along these lines in [17].

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